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A stochastic differential equation model for the shoreline evolution

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Contents

1 Introduction

- Physical factors controlling the shoreline evolution are the wave climate, existing shoreline position, sediment supplies and properties. Since most of these factors cannot be determined precisely, the dynamical response of the shoreline over time has to be treated as a time-dependent stochastic system.
- Various probabilistic methods have been proposed some of which deal with the variability of hydrodynamic input while others account for the variability in model parameters.
- Earlier probabilistic modelling (Monte Carlo sampling with or without chronology effects) => Direct solving modelling (time-averaged) =>Process-based Fokker-Planck model (time-dependent)

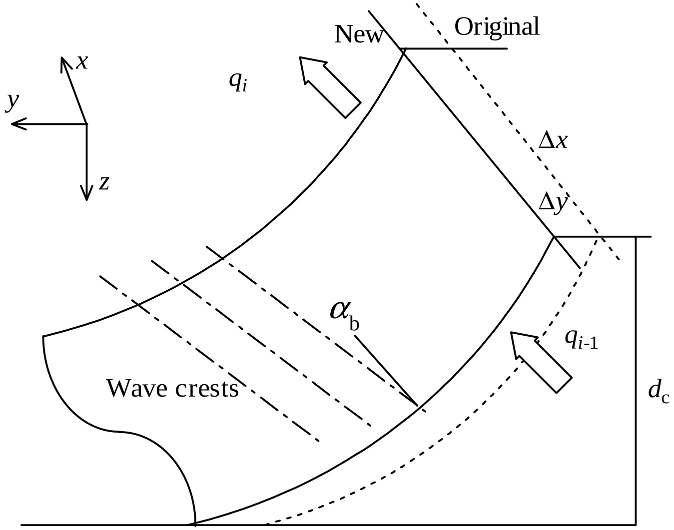
1 Past work and techniques used

categories	Advantages	Disadvantages
Monte Carlo sampling	Easy to implement Vrijling and Mejer (1992) Dong and Chen (1999) Hall et.al (2002)	1) extremely large computational resources, especially for accurately computing the low-probability of a distribution. 2) Chronology effect
Time-averaged statistic dynamic approach	Direct solution Reeve & Spivack (2004)	1) Forcing reflect general cumulative effects but single 2) No direction information on PDF(probability density function)
Fokker-Planck model	Direct solution Time dependent Capturing instantaneous PDF	1) Normal distribution 2) Can other distribution by Liouville model

2 One-line coastal recession model

Only longshore transport component considered as a contribution
The general equation for the deterministic process

$$\frac{\partial y(x,t)}{\partial t} = -\frac{1}{D_c}\frac{\partial q_l(x,t)}{\partial x} \qquad (1)$$



$$q_l(x,t) = q_{l0}(x,t)\sin\alpha_b(x,t)\cos\alpha_b(x,t) \qquad (2)$$

$$q_{l0} = (E_b C_{gb})a_1$$

$$a_1 = \frac{k_1}{16(\rho_s/\rho_w - 1)(1-n)g}$$

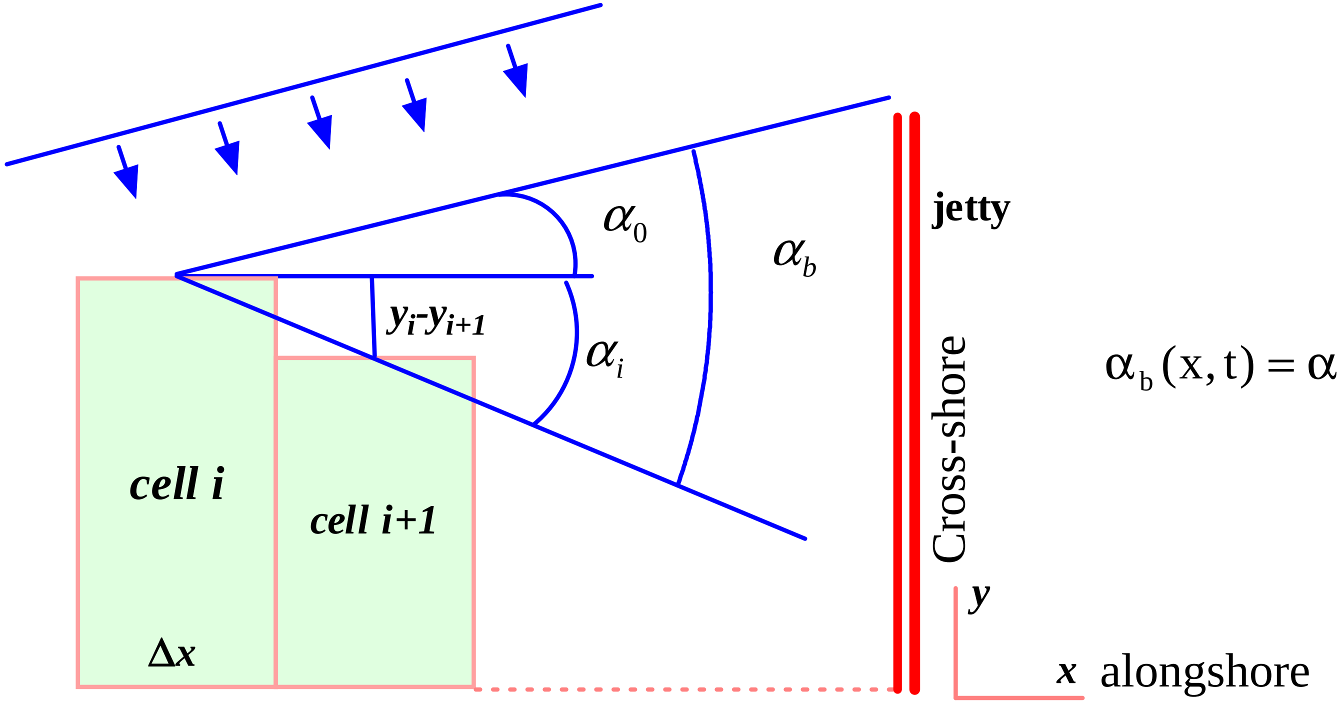
$k_1 = 0.41 \quad \rho_s/\rho_w = 2.65$

$n = 0.4 \quad a_1 = 0.247 \quad h \approx 0.78H$

$$E_b = \frac{1}{8}\rho_w g H^2 \qquad C_{gb} = \frac{1}{2}\sqrt{gh}$$

$$q_{l0} = H^{5/2}\frac{1}{16}\rho_w g^{3/2}\gamma^{1/2}a_1$$

2 Parameters of coastal recession model



$$\alpha_b(x,t) = \alpha_0(x,t) - \arctan\left(\frac{\partial y}{\partial x}\right)$$

(3)

In general, q_{l0} α_0 α_b all vary in time and alongshore

$$\frac{\partial y(x,t)}{\partial t} = -\frac{1}{D_c}\frac{\partial q_l(x,t)}{\partial x}$$

Nume

Numerical scheme

3 SDE model formulation

Based on discrete time observations, random time varying process

$$\begin{cases} \dfrac{\mathrm{d}y(x,t)}{\mathrm{d}t} = f(y,x,H) \\ y(x,0) = y_0 \end{cases} \qquad (6)$$

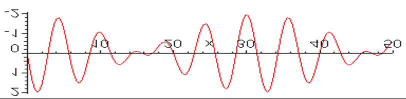
With a random variables wave height and boundary condition

$$H^{5/2} = H_{\mathrm{w}} \quad H_{\mathrm{w}} = K_{\mathrm{m}} + W(t) \qquad D = C_v K_{\mathrm{m}} \quad 0 \le C_v \le 1$$

Which is a **standard random differential equation** with a input term and random initial condition

W_t is a Brownian motion in physic, called by Wiener process

$\dfrac{dW_t}{dt}$ denotes for a white noise The variance is D



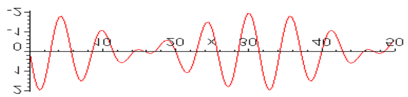
3 SDE model formulation – Cont.

Substitute one-line model to random differential equation, then **a random process model of shoreline position** can be given

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = -[K_\mathrm{m} + W(t)]\phi(y,t); \qquad W(t) \sim N(0,D) \qquad (7)$$

$$\phi(y,t) = \frac{1}{16}\rho_\mathrm{w} g^{3/2}\gamma^{1/2} a_1 (\sin 2\alpha_\mathrm{b}{}^{i-1} - \sin 2\alpha_\mathrm{b}{}^{i}) / (\Delta x d_\mathrm{c})$$

Transfer function



Which can be reformulated as

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = \phi(y,t)K_\mathrm{m} + G(y,t)W(t) \qquad (8)$$

here $G(y,t) = \phi(y,t)$

Drift term (mean) & random diffusion (variations)

3 SDE – Ito equation

Above equation is equivalent (see Soong, 1973) to the stochastic **Ito** equation

$$\mathrm{d}y(x,t)=\phi(y,t)K_{\mathrm{m}}\mathrm{d}t+G(y,t)\mathrm{d}B(t) \qquad (9)$$

dt

$$E\{\mathrm{d}B(t)\}=0 \qquad E\{[\mathrm{d}B(t)]^2\}=2D\mathrm{d}t$$

which D is the entry of the covariance parameter of the white noise process

The solution scheme of Ito equation (9) can be given by Monte Carlo sampling

ITO

$$y_j=y_{j-1}+\phi(y,t)K_{\mathrm{m}}\Delta t+G(y,t)[(W(\tau_j)-W(\tau_j))]$$

3 SDE – Fokker Planck Equation

It is known that system described by equation (9) has a transition PDF (refer to Soong, 1973), satisfies the **Fokker-Planck equation**

$$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left[\phi(y,t)K_{\mathrm{m}}\, p\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[(GDG^T)\, p\right] \qquad (10)$$

Therefore the problem of determining the probability density is transformed into the problem of solving above partial differential equation.

initial and boundary condition

normalization condition

$$\int_{y_{\min}}^{y_{\max}} p(y,t)\mathrm{d}y = 1$$

$$p(y,t_0) = p_0(y)$$

$$p(y_{\max},t) = 0$$

$$p(y_{\min},t) = 0$$

3 Solution of Fokker Planck

The Fokker-Planck equation is a deterministic partial differential equation for a nonlinear Ito equation problem for which it is difficult to obtain an exact theoretical solution.

Various numerical schemes can been used,like, finite element methods, finite difference methods, and path integral methods

The backward in time and central in space method is obtained by using a second-order implicit difference approximation. For n to n+1 time step and j cell (Crank-Nicholson time integration), we have

$$\frac{p_j^{n+1} - p_j^n}{\Delta t} = \theta F_j^{n+1}(p) + (1-\theta) F_j^n(p)$$

$$F_j^n(p) = \frac{D(G_{j+1}^n)^2 p_{j+1}^n - 2D(G_j^n)^2 p_j^n + D(G_{j-1}^n)^2 p_{j-1}^n}{\Delta y^2} + \frac{\phi_{j+1}^n p_{j+1}^n - \phi_{j-1}^n p_{j-1}^n}{2\Delta y} K_{\mathrm{m}}$$

the matrix form

$$[K]\{p\} = [F]$$

FP

3 Solution on Fokker Planck

$$k_{i,j-1} = -\frac{\theta\phi_i^n}{2\Delta y}K_{\mathrm{m}} - \frac{\theta D(G_i^n)^2}{\Delta y^2}$$

$$k_{i,j} = \frac{1}{\Delta t} + \frac{2\theta D(G_i^n)^2}{\Delta y^2} \qquad k_{i,j} = 0 \qquad j \neq i-1, i, i+1 \qquad (13)$$

$$k_{i,j+1} = \frac{\theta\phi_i^n}{2\Delta y}K_{\mathrm{m}} - \frac{\theta D(G_i^n)^2}{\Delta y^2}$$

$$F_i = \frac{p_i^n}{\Delta t} - (1-\theta)\frac{\phi_{i+1}^n p_{i+1}^n - \phi_{i-1}^n p_{i-1}^n}{2\Delta y}K_{\mathrm{m}} + (1-\theta)D\frac{(G_{i+1}^n)^2 p_{i+1}^n - 2(G_i^n)^2 p_i^n + (G_{i-1}^n)^2 p_{i-1}^n}{\Delta y^2}$$

above tri-diagonal matrices can be fairly efficiently solved utilizes Thomas algorithm by row operation (see Wang and Anderson, 1982)

3 Results on FPE – Moments

In the solution process, y_m and $_{ys}$ the corresponding mean and deviation of shoreline position can be calculated by using an integration method.

1st moment --- Mean

$$y_{\mathrm{m}}(t) = \int_{y_{\min}}^{y_{\max}} y_i(y,t)\, p(y,t)\mathrm{d}y \qquad (14)$$

2nd moment --- Deviation

$$y_{\mathrm{s}}(t) = \sqrt{\int_{y_{\min}}^{y_{\max}} y_i(y,t)^2\, p(y,t)\mathrm{d}y} \qquad (15)$$

3 Summary on algorithm

Drift $$u(y) = K_{\mathrm{m}}\phi(y,t) = K_{\mathrm{m}}\frac{1}{16}\rho_{\mathrm{w}}g^{3/2}\gamma^{1/2}a_1(\sin 2\alpha_{\mathrm{b}}{}^{i-1} - \sin 2\alpha_{\mathrm{b}}{}^{i})/(\Delta x d_{\mathrm{c}})$$

Diffusion $$\sigma^2(y) = G(x,y,t)W(t)$$

SDE Ito $$\mathrm{d}y(x,t) = \phi(y,t)K_{\mathrm{m}}\mathrm{d}t + G(y,t)\mathrm{d}B(t)$$

FPE $$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left[\phi(y,t)K_{\mathrm{m}}p\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[(GDG^T)p\right]$$

3 Soluti

mean value and variance of wave height, initial distribution of shoreline position and its boundary condition and range, initial configuration of shoreline.

time ste

each cell

Computing the shoreline position by process-based one line model, then calculate the breaking angle and long-shore transport rate

Predicting the function value of G and ϕ , and then determine the tri-diagonal matrices and source term, solving the FP equations to get PDF by Thomas algorithm

Integrating mean value of the shoreline position for using in next time step

next cell

The uniformly meshing strategy of selecting points is employed.

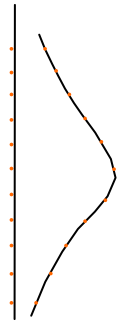
next timestep and output

4 results

test cases to illustrate the concepts and capabilities of the method

Constant
a1=0.247/ alfa=15 degree
Increments
dx=25m/dt=0.1day
Ncell=100/nstep=3650
Dc=10.5m
Variables
Km=0.365/Cv=0.15
y0=0/deviation of y0=2.5m

Considering the maximum deposition distance and algorithm precision of the values of the **PDF** at the nodes
dy=0.5m/ 1501nodes
ymax=650m/ymin=-100m



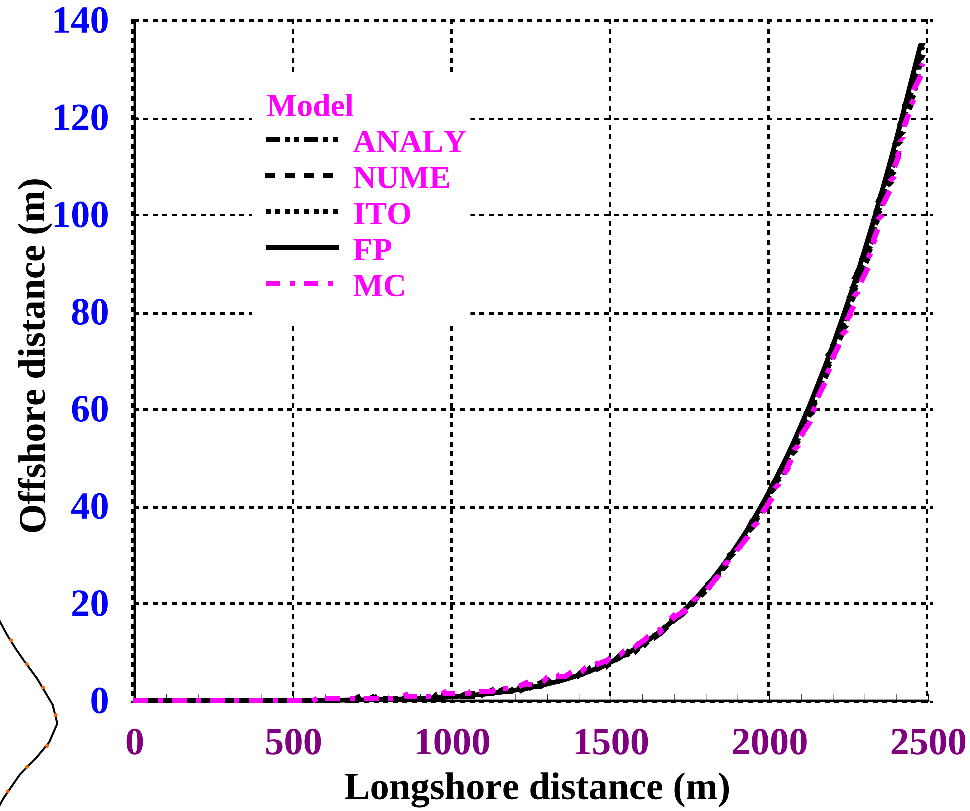


Fig. 2 Comparison of shoreline changing y_m by various model

4 results

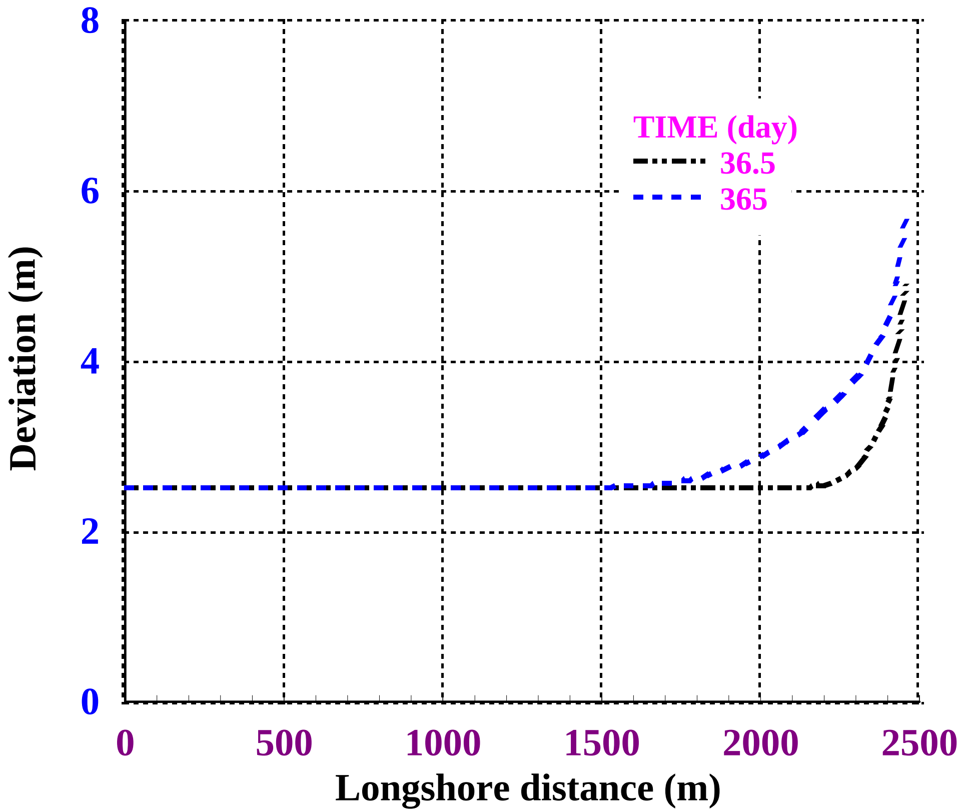
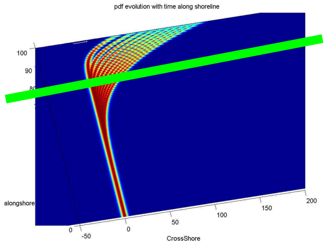


Fig. 3 Distribution of deviation of shoreline location y_s

4 results



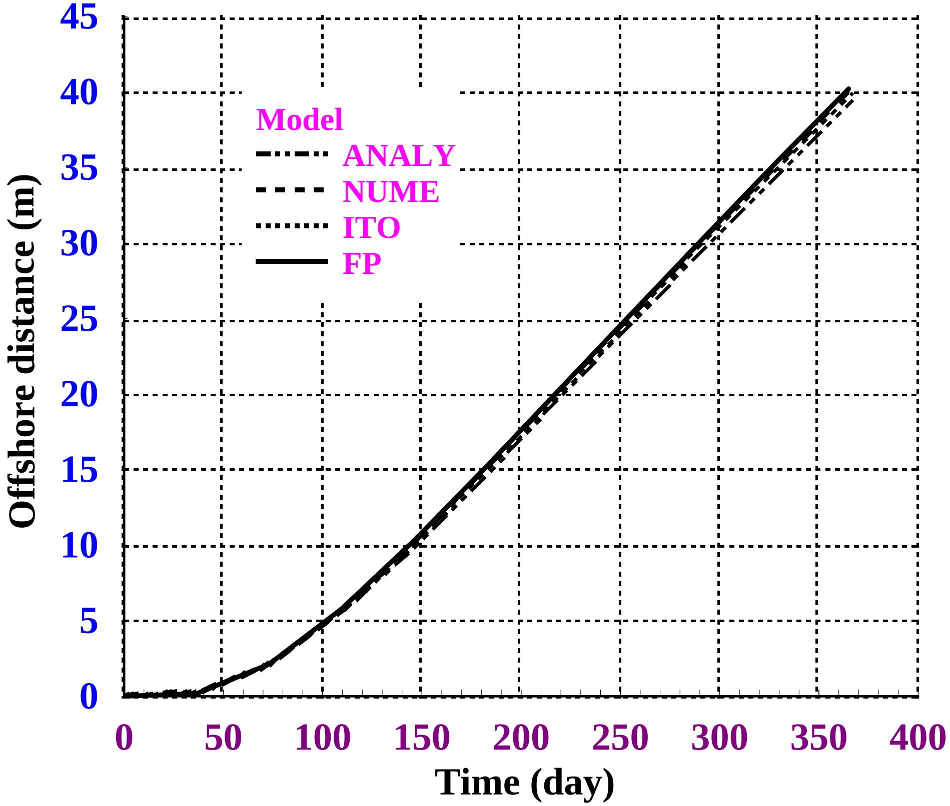


Fig. 4 shoreline changing with time at cell 80 by various model

4 results

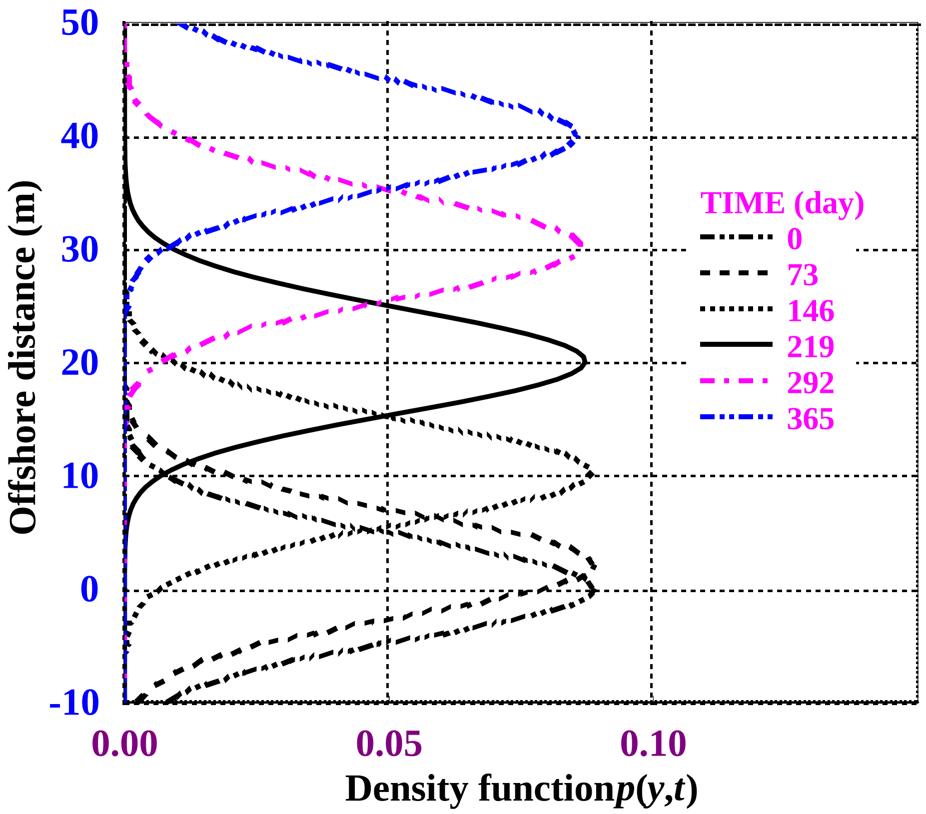


Fig. 5 Variation of PDF over time

4 Results pdf evolution in 3d space

Like the alluvial fan stretching down to the distance

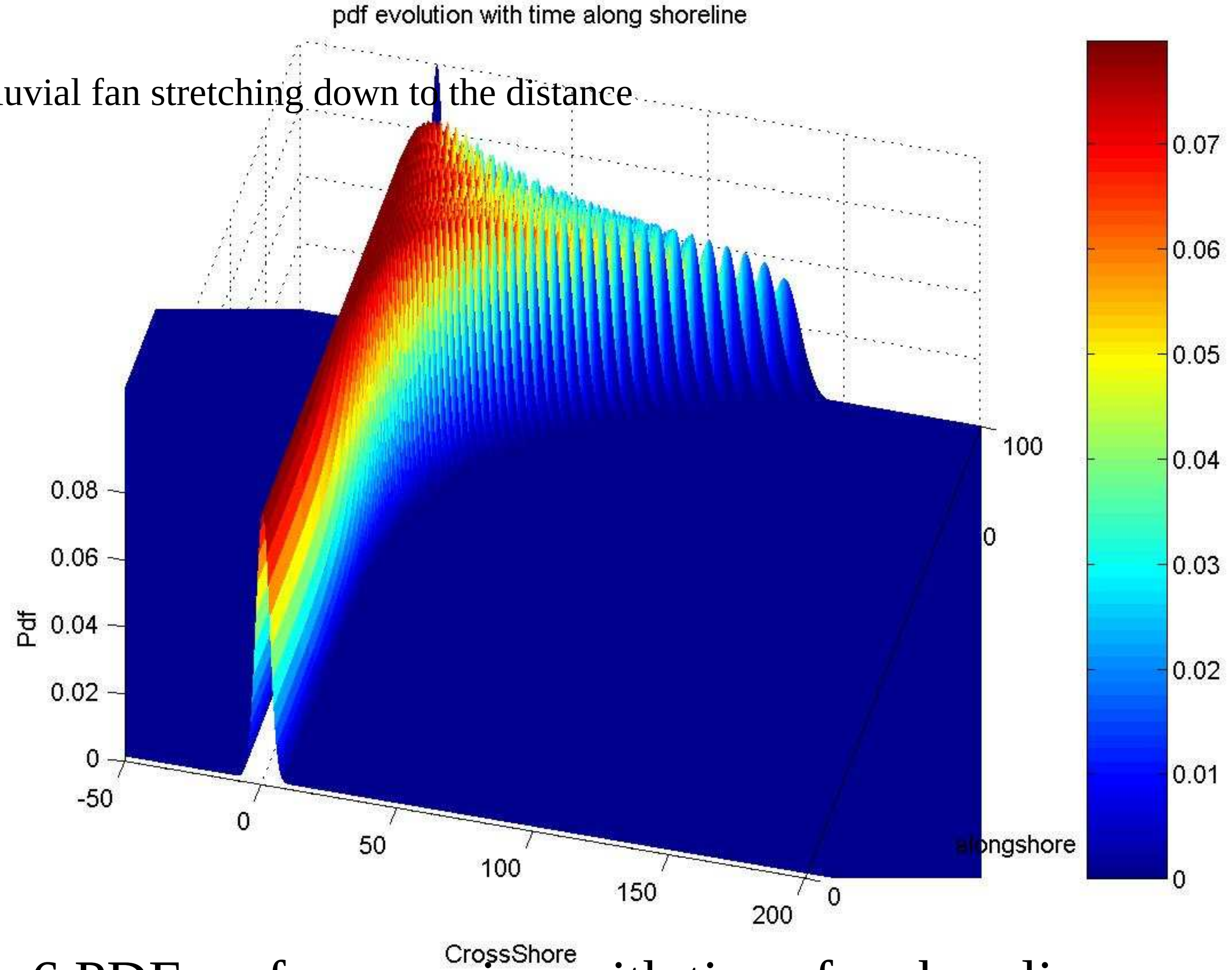


Fig. 6 PDF surface varying with time for shoreline

4 Results pdf in 3d space at 80th cell

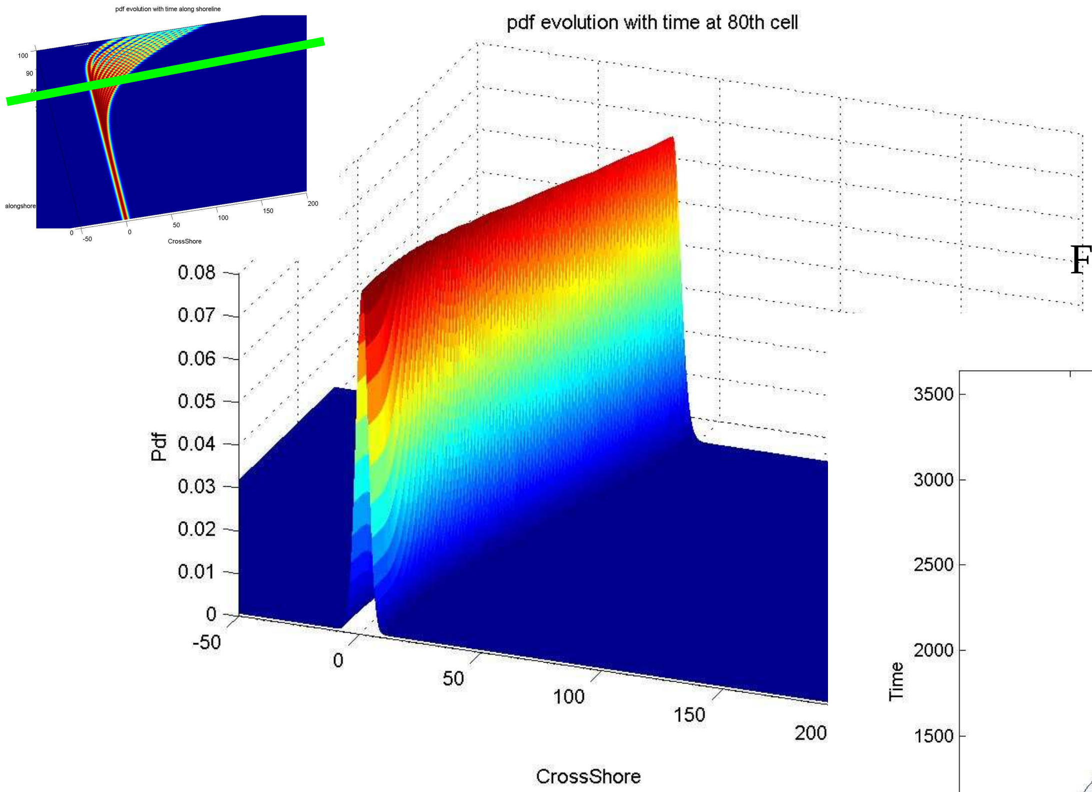
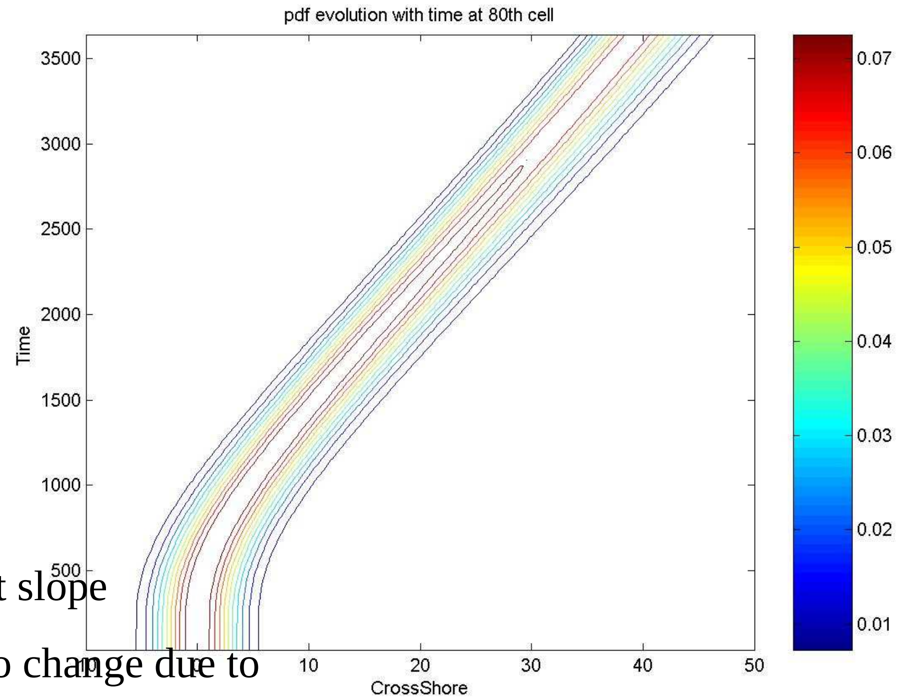


Fig. 7a PDF surface varying with time at cell 80

Fig. 7b The coutours of PDF at 80th



Seems like water flowing in a river with a gradient slope

The flowin the “river” is not stationary, initially no change due to no deposition at the beginning then drift constantly over time

4 results – parametric sensitivity

Km=0.156, 0.25, 0.375, and 0.53

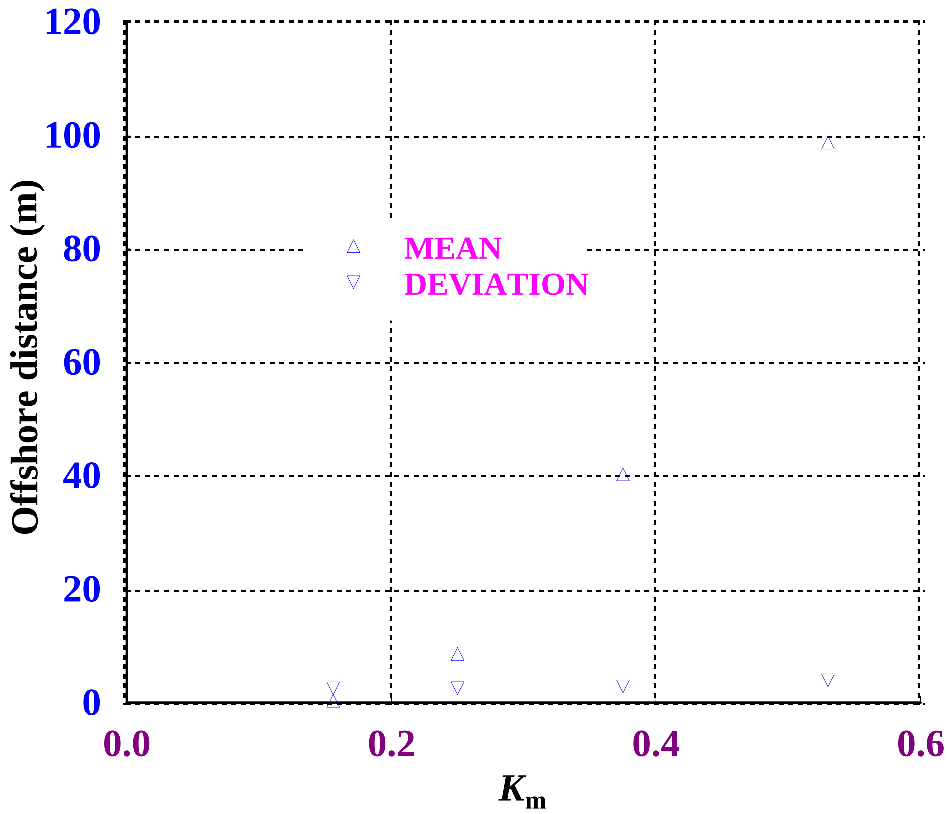


Fig. 8 Mean value and deviation of shoreline position at various Km

4 results - parametric sensitivity

Cv=0.05, 0.15, 0.30 and 0.45

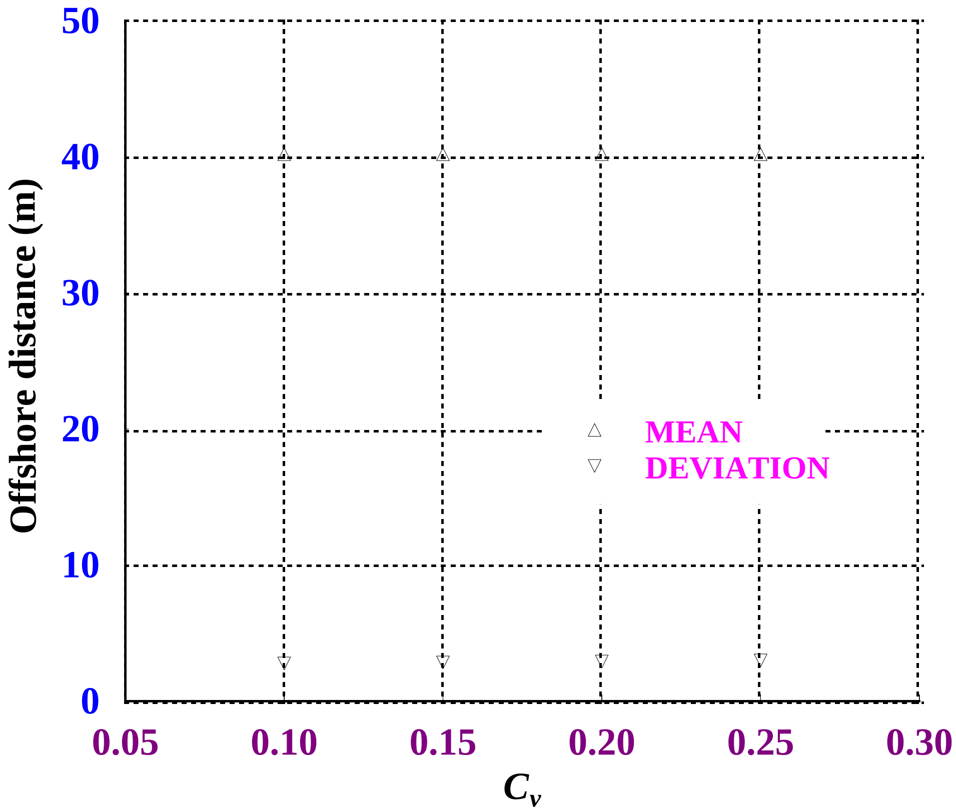


Fig. 9 Changes of mean and deviation of shoreline position with variation coefficient Cv of wave height

4 results - parametric sensitivity

input a serials deviation of , y_0 varying from 0 to 6

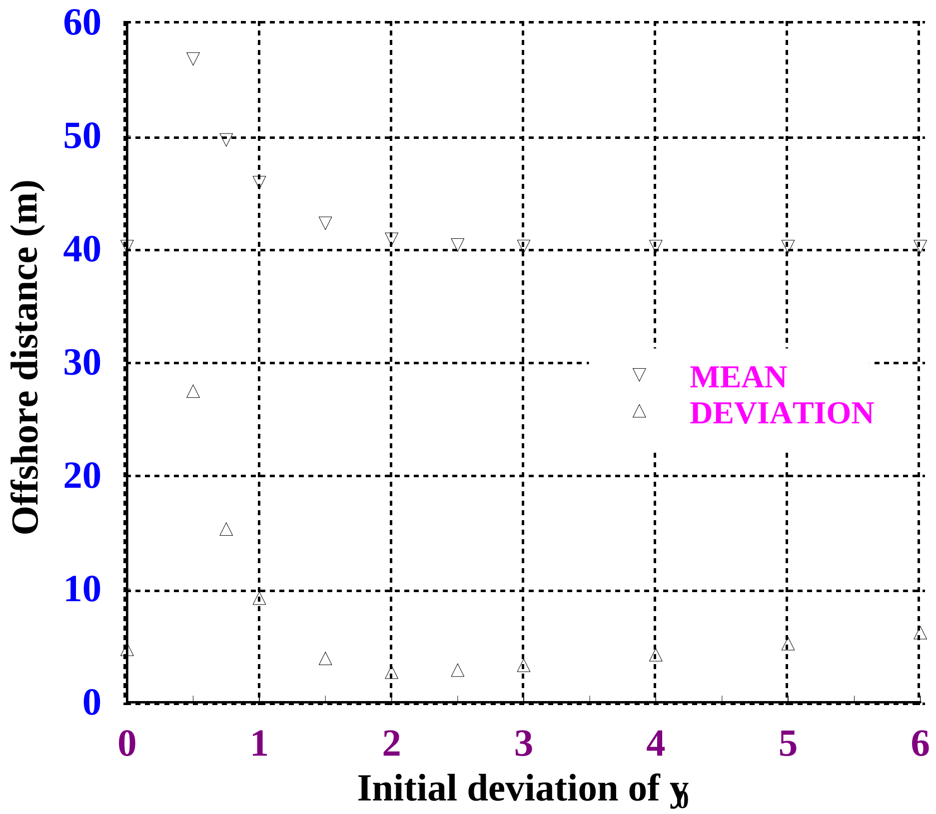


Fig. 10 Changes of y_m and y_s with y_{s0}

4 Results – time dependent reliability

The dynamic reliabilities of the shoreline at cell 80 are evaluated with a thresholds is 40.0m. Listed in Table 1 are the reliabilities with different time by integration method and U.L.S. method.

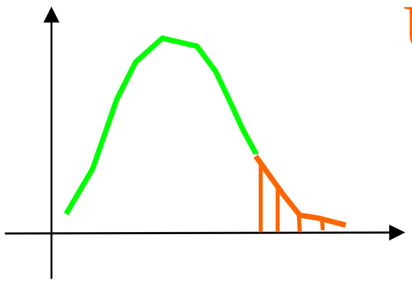
Obviously the reliability will decrease over time due to accretion and failure will happen near the final.

Actually, normal distribution of input variables do generally not lead to a normal distribution output of the coastline position (Vrijling 1993, Hall et al., 2002).

By the way, the proposed method is quite more efficient than the U.L.S..

Integration $R = P_r\{y_L \le Y(\tau) \le y_U, \tau \in [0,T]\}$

$$R = \int_{\Omega_s} p_{Y\max}(y)\mathrm{d}y$$



U.L.S. $P_\mathrm{f} = \iint_{r<s} f_R(\mathrm{r}) f_S(s)\mathrm{d}r\mathrm{d}s$

$$p_\mathrm{f}\{Y > 40.0\} = \Phi\{(\mu_Y - 40)/\sigma_Y\}$$

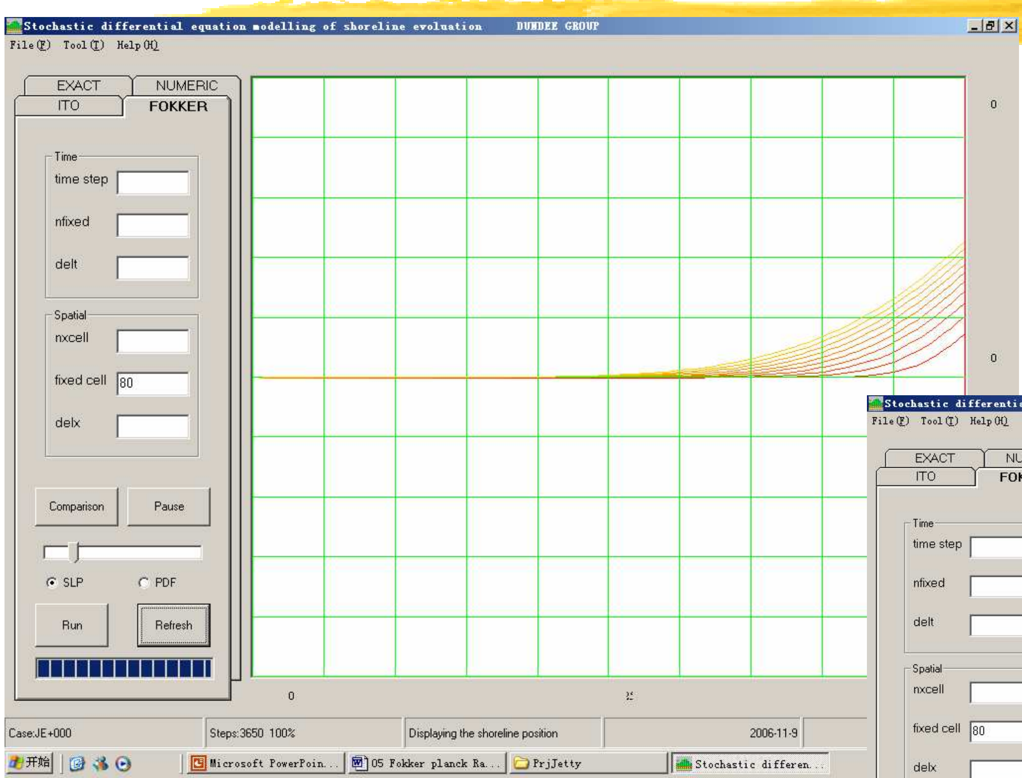
$$R = 1 - p_\mathrm{f}$$

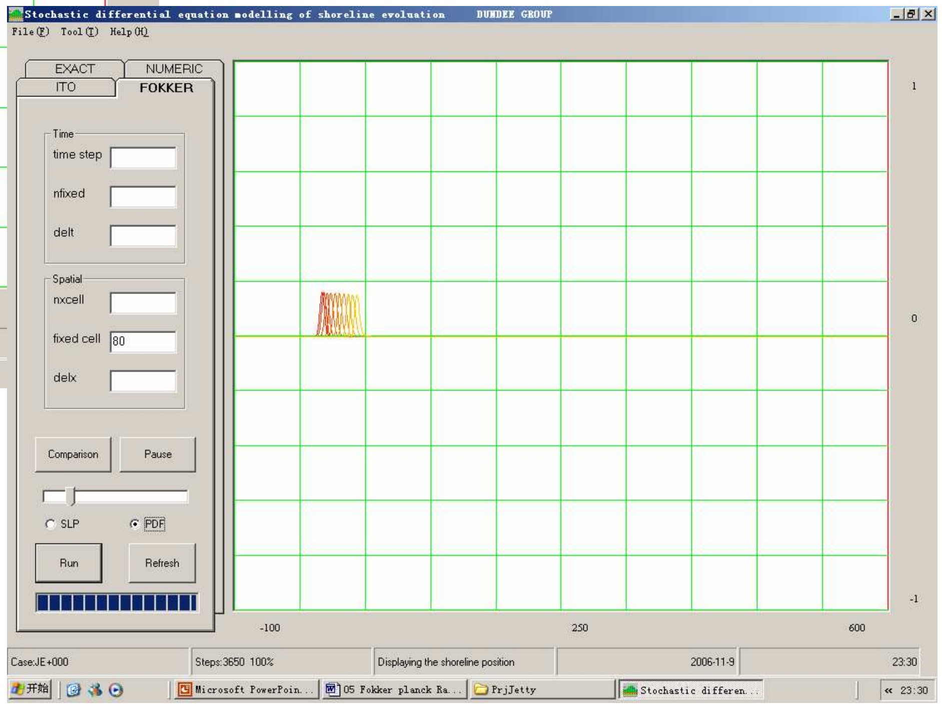
$\Phi\{\ \}$ is standard normal distribution

Table 1. The dynamic reliabilities

Time (days)	Reliability	
	Integration	U.L.S.
292	1.0	1.0
328.5	.895	0.902
365	0.261	0.2716

Demonstration software





5 Conclusions

- Shoreline random evolution can in principle be treatable using the formalism and techniques that are available for solving stochastic partial differential equations (SPDE) , probabilistic solution of shoreline was transferred to be a deterministic differential equation

- Comparing the results with some other solutions by deterministic or stochastic approach, very similar trend and reasonable distributions have been predicted. It is shown that the proposed random models can reflect the effects of uncertainties of parameters and boundary condition on the shoreline position and with further extension can be used for predictive purpose

5 Conclusions

- The statistics of coastline position is not a stationary with respect to either space or time, which is unable to predict the process wholly by conventional deterministic model.
- Having determined the PDF of the shoreline position, which can be employed to determine the time-dependent reliability or risk of shoreline position by supposing exceeding a certain distance using the uncertainty analysis with varied wave climate.
- If the wave angle were to be treated as a random variable as well, no linear relationship in stochastic differential equation of shoreline exists and a FP model for the shoreline would not be possible. In this case, a Liouville model has to employ and can handle with non-Gaussian distribution problem as well.

Acknowledgements

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- Thanks to Prof. Ping Dong and other colleagues in Dundee geotechnical group

The end

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Thanks